

Abelian vortices from Sinh–Gordon and Tzitzeica equations

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Abstract

It is shown that both the sinh–Gordon equation and the elliptic Tzitzeica equation can be interpreted as the Taubes equation for Abelian vortices on a CMC surface embedded in $\mathbb{R}^{2,1}$, or on a surface conformally related to a hyperbolic affine sphere in \mathbb{R}^3 . In both cases the Higgs field and the $U(1)$ vortex connection are constructed directly from the Riemannian data of the surface corresponding to the sinh–Gordon or the Tzitzeica equation. Radially symmetric solutions lead to vortices with a topological charge equal to one, and the connection formulae for the resulting third Painlevé transcendents are used to compute explicit values for the strength of the vortices.

1 Introduction

The Abelian Higgs model is a relativistic field theory of a complex scalar field ϕ coupled to an Abelian gauge field a on a 2+1 dimensional space–time $\Sigma \times \mathbb{R}$. Here Σ is a two–dimensional surface with a Riemannian metric g and the space–time metric is $g - dt^2$. This model admits topological solitons called vortices [13] relevant in the theory of thin superconductors [1].

At the ‘critical coupling’ the vortices arise as solutions to the first order Bogomolny equations. These Bogomolny equations are in general not integrable and the form of static finite energy solutions is not known. This sets the vortices apart from other topological solitons (lumps, monopoles, Yang–Mills instantons), where the Bogomolny equations are integrable and large families of explicit solutions exist [6]. The one exception is when the the surface (Σ, g) is a hyperbolic space with a metric of constant curvature $-1/2$. In this case the Bogomolny equations reduce to the Liouville equation whose solutions are known [20, 18, 12].

The aim of this note is to show that there are two more integrable cases. In both cases the Higgs field and the connection are constructed directly

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from the Riemannian data of the surface (Σ, g) . Thus, unlike the hyperbolic vortex, the resulting solutions are isolated points in the moduli space: changing the parameters of a vortex would correspond to changing the background metric on Σ . The two integrable cases correspond to (Σ, g) being a surface of constant mean curvature in $\mathbb{R}^{2,1}$ and a surface conformal to a hyperbolic affine sphere in \mathbb{R}^3 respectively. In both cases the norm of the Higgs field is given by a power of the conformal factor in the metric g , and the $U(1)$ connection is identified with a Levi-Civita connection one-form of g . The vortex number is given by the Euler characteristics of $T\Sigma$. In the first case the Bogomolny equations reduce to an elliptic sinh-Gordon equation, and in the second case they reduce to an elliptic Tzitzeica equation. Both sinh-Gordon and Tzitzeica equation arise as symmetry reductions of anti-self-dual Yang-Mills equations in \mathbb{R}^4 and thus are integrable [5, 6]. The radial solutions of sinh-Gordon and Tzitzeica equations are characterised by 3rd Painlevé transcendents with parameters $(0, 0, 1, -1)$ and $(1, 0, 0, -1)$ respectively. Using the known connection formulae [14, 10, 11] for these transcendents relating asymptotic of the solutions at 0 and ∞ we are able to find analytic expressions for strengths of the corresponding one-vortex solutions.

2 Taubes equation

Let L be a Hermitian complex line bundle over a Riemannian surface (Σ, g) and let ω be a Kähler form corresponding to some choice of a complex structure on Σ . We define vortices as finite energy solutions of the Bogomolny equations

$$\bar{D}\phi = 0, \quad F = \frac{1}{2}\omega(1 - |\phi|^2),$$

where the Higgs field ϕ is a global C^∞ section of L , \bar{D} is the anti-holomorphic part of the covariant derivative of a $U(1)$ connection a on L compatible with the Hermitian structure of L , and finally F is the curvature of a . The energy functional is given by

$$E[a, \phi] = \frac{1}{2} \int_{\Sigma} \left(|D\phi|^2 + |F|^2 + \frac{1}{4}(1 - |\phi|^2)^2 \right) \text{vol}_{\Sigma}.$$

Let $z = x + iy$ be a local holomorphic coordinate on Σ such that the metric takes the form

$$g = \Omega dzd\bar{z}, \quad \text{where} \quad \Omega = \Omega(z, \bar{z}).$$

Choosing a trivialisation of L such that $a = a_z dz + \bar{a}_z d\bar{z}$, the curvature is given by the magnetic field $B = \partial_x a_y - \partial_y a_x$ and the Bogomolny equations become

$$\frac{\partial \phi}{\partial \bar{z}} = i a_{\bar{z}} \phi \tag{1}$$

and

$$B = \frac{\Omega}{2}(1 - |\phi|^2). \tag{2}$$

We solve the first equation for a and set $\phi = \exp(h/2 + i\chi)$, where $h, \chi : \Sigma \rightarrow \mathbb{R}$. The second equation then yields the Taubes equation [19]

$$\Delta_0 h + \Omega - \Omega e^h = 0, \quad \text{where} \quad \Delta_0 = 4\partial_z \partial_{\bar{z}}. \quad (3)$$

This is valid outside small discs enclosing the logarithmic singularities of h . The vortices are centred at points in Σ where the Higgs field vanishes or equivalently where $h \rightarrow -\infty$. The vortex number is defined to be

$$\frac{1}{2\pi} \int_{\Sigma} B \, \text{vol}_{\Sigma},$$

and an N -vortex solution centred at $z = z_0$ has an expansion of the form

$$h \sim 2N \log |z - z_0| + \text{const} + \frac{1}{2} \bar{b}(z - z_0) + \frac{1}{2} b(\bar{z} - \bar{z}_0) + \dots \quad (4)$$

as $|z| \rightarrow 0$, where the coefficients expansion depend on the position of the vortex. The moduli space of solutions to (1) and (2) with vortex number N is a manifold of real dimension $2N$.

3 Vortices from sinh–Gordon equation

Taking $\Omega = \exp(-h/2)$ in the Taubes equation gives the elliptic sinh–Gordon equation

$$\Delta_0 \left(\frac{h}{2} \right) = \sinh \left(\frac{h}{2} \right). \quad (5)$$

This gives an interpretation of the metric g as an isolated vortex. The magnetic field and the Higgs field have an intrinsic geometric interpretation as the (Hodge dual of) the Riemann curvature two-form and the (inverse of) conformal factor with a complex phase. The Higgs field vanishes at the position of the vortex and at this point the conformal factor becomes infinite. The norm of the Higgs field $|\phi|^2$ tends to its asymptotic value 1 as $x^2 + y^2 \rightarrow \infty$.

The first Bogomolny equation (1) asserts that the $U(1)$ connection a is equal to the Levi–Civita connection one-form of g . This is true in some trivialisation of the underlying line bundle L identified with the tangent bundle $T\Sigma$. The holonomy group $SO(2)$ of the Levi–Civita connection is then identified with the gauge group $U(1)$. The Chern number of L becomes the Euler characteristic of $T\Sigma$.

The second Bogomolny equation (2) is the sinh-Gordon equation (5). It imposes conditions on the metric g which are equivalent to the statement that the background surface (Σ, g) is a space-like immersion with constant mean curvature (CMC) in the flat Lorentzian three-space $\mathbb{R}^{2,1}$. In this context equation (5) arises from the Gauss–Codazzi equations [3, 4, 21].

To see the explicit relations between the Riemannian data of g and the vortex, set $h = -2u$ so that

$$\begin{aligned}\Delta_0(u) &= \sinh(u), \quad g = e^u dz d\bar{z}, \quad \text{and} \\ B &= \Delta_0(u), \quad \phi = e^{-u+i\chi}, \quad a = i(\bar{\partial} - \partial)u - d\chi,\end{aligned}\tag{6}$$

where $\partial = dz \otimes \partial/\partial z$. Choose a spin-frame $\mathbf{e}^1 = e^{u/2}dx, \mathbf{e}^2 = e^{u/2}dy$ such that $g = \delta_{ij}\mathbf{e}^i \otimes \mathbf{e}^j$. The connection and curvature forms of g can be read off from the Cartan structure equations and are given by

$$\begin{aligned}\Gamma^1_2 &= \frac{1}{2}(u_y dx - u_x dy) = -\frac{1}{2}a - \frac{1}{2}d\chi, \\ R^1_2 &= -\frac{1}{2}(u_{xx} + u_{yy})dx \wedge dy = -\frac{1}{2}Bdx \wedge dy.\end{aligned}$$

A simple solution to (5) with $h = -2u$ is

$$u = 4 \tanh^{-1} \left(\exp \left(\frac{(z - z_0)e^{-i\alpha}}{2} + \frac{(\bar{z} - \bar{z}_0)e^{i\alpha}}{2} \right) \right),\tag{7}$$

where α is a real parameter. This is an analytic continuation of the one-kink soliton of the better known sine-Gordon equation. Setting $\alpha = 0, z_0 = 0$ gives the metric on the surface of revolution

$$g = \tanh(x/2)^{-2}(dx^2 + dy^2).$$

Expanding this around $x = 0$ gives the half plane metric

$$\frac{1}{4} \frac{dx^2 + dy^2}{x^2}.$$

The expansion of h around 0 is $4 \log|x| + \dots$ which however can not be interpreted as a two-vortex solution as B doesn't vanish in the y direction and the integral is infinite.

To reinterpret the CMC surface (6) as a vortex we need to construct a solution of $\Delta_0 u = \sinh(u)$ such that

$$u \rightarrow 0 \quad \text{as} \quad r \rightarrow \infty$$

and $h = -2u$ behaves like (4) near $r = 0$. We shall look for radial solutions of the form $u = u(r)$. The sinh-Gordon equation reduces to an ODE

$$u_{rr} + \frac{1}{r}u_r = \sinh u.\tag{8}$$

This is equivalent to the Painlevé III ODE with special values of parameters. For large r we approximate $\sinh u = u$, and obtain the modified Bessel equation of order zero, so

$$u \sim 4\lambda K_0(r) \quad \text{as} \quad r \rightarrow \infty,\tag{9}$$

for some λ which needs to be determined by the initial condition at $r = 0$. For small r we instead get the Liouville equation

$$u_{rr} + \frac{1}{r}u_r = \frac{1}{2}e^u \quad (10)$$

which can be solved exactly. Thus

$$u \sim -2\sigma \log(r) + \text{const} + O(r) \quad \text{as } r \rightarrow 0 \quad (11)$$

for some constant σ (the overall multiples 2 and 4 in these asymptotic have been chosen for convenience). It is known [9] that if $0 < \lambda < \pi^{-1}$ there exists a solution to the radial sinh–Gordon equation whose only singularity is at 0. To find how λ depends on σ we shall refer to the results about the Painlevé III asymptotics. Setting $u = -2 \ln w$ and $r = 2\rho$ in (10) yields

$$w_{\rho\rho} = \frac{(w_\rho)^2}{w} - \frac{w_\rho}{\rho} + w^3 - \frac{1}{w},$$

which is the Painlevé III equation with parameters $(0, 0, 1, -1)$.

The connection formulae relating the asymptotic solution to PIII at $r = 0$ and $r = \infty$ have been derived in [14]. Using the asymptotic formula for the modified Bessel function

$$K_0(r) = \sqrt{\frac{\pi}{2}} \frac{1}{\sqrt{r}} e^{-r} \quad (12)$$

valid for large r and applying the results¹ of [14] we find that $h = -2u$ is of the form

$$\begin{aligned} h(r) &\sim 4\sigma \ln r + 4 \ln \beta + \frac{1}{\beta^2} r \quad \text{for } r \rightarrow 0 \\ &\sim -8\lambda K_0(r) \quad \text{for } r \rightarrow \infty \end{aligned} \quad (13)$$

with the connection formulae

$$\sigma(\lambda) = \frac{2}{\pi} \arcsin(\pi\lambda), \quad \beta(\lambda) = 2^{-3\sigma} \frac{\Gamma((1-\sigma)/2)}{\Gamma((1+\sigma)/2)},$$

where Γ is the gamma function. This is valid for $0 \leq \lambda \leq \pi^{-1}$. The asymptotic expansion (4) implies that the N -vortex solution has $\sigma = N/2$. Thus the asymptotic connection formulae are valid only if $N = 1$, and there exists a one-vortex solution with $\sigma = 1/2$ and

$$\lambda = \frac{\sqrt{2}}{2\pi}, \quad \beta = 2^{-3/2} \frac{\Gamma(1/4)}{\Gamma(3/4)}.$$

¹We need formula (1.10) from this reference with $u(r) = -2 \log(w(r/2))$, $\nu = 0$ and expressions (1.11) and (1.12) from *Theorem 3*.

In the particle interpretation of Speight [16], vortices, when viewed from a large distance, behave as point particles with the strength given by a coefficient of the Bessel function in the asymptotic expansion (13) of h . Thus the strength of the sinh–Gordon vortex is

$$\frac{4\sqrt{2}}{\pi} \sim 1.80.$$

For comparison, the strength of a plane vortex can not be calculated analytically. The approximate value 1.68 has been found numerically [13].

Near $r = 0$ the metric g in (6) with $z = r \exp i\theta$ takes the form

$$g = \frac{4}{\beta^2} (dR^2 + \frac{1}{4} R^2 d\theta^2), \quad \text{where} \quad R = r^{-1/2}.$$

Thus close to $r = 0$ we obtain a flat metric with conical singularity at the origin and the 1–vortex deficit angle is π . The corresponding CMC surface is an analogue the Smyth CMC surface [17].

Calculating the vortex number for the solution with asymptotic (9) and (11) yields

$$\begin{aligned} \frac{1}{2\pi} \int_{\mathbb{R}^2} B \, dx dy &= \frac{1}{2\pi} \lim_{r_2 \rightarrow \infty} \lim_{r_1 \rightarrow 0} \int_0^{2\pi} \int_{r_1}^{r_2} \frac{1}{r} (ru_r) \, r dr d\theta \\ &= 4\lambda \lim_{r_2 \rightarrow \infty} (-r_2 K_1(r_2)) - \lim_{r_1 \rightarrow 0} (-2\sigma + O(r_1)) \\ &= 2\sigma \end{aligned}$$

as the modified Bessel function $K_1(r)$ decays exponentially at ∞ . Thus the vortex number is 1 if $\sigma = 1/2$ as expected.

3.1 Vortices from Tzitzeica equation

There is another possible choice of the conformal factor in (3) which leads to an integrable equation. Setting $h = 3u$ and choosing $\Omega = \exp(-2u)$ leads to the elliptic Tzitzeica equation

$$\Delta_0 u + \frac{1}{3} (e^{-2u} - e^u) = 0.$$

There are several versions of the Tzitzeica equation which depend on the relative signs between the exponential terms [7]. The one above corresponds to hyperbolic affine sphere in \mathbb{R}^3 , [22]. with the Blaschke metric $g_B = e^{3u} g$.

Assuming that $u = u(r)$ and setting (similar reductions have been accomplished in [10, 7])

$$u(r) = \ln(w(r)) - \frac{1}{2} \ln r + \frac{1}{4} \ln\left(\frac{27}{4}\right), \quad r = \frac{3\sqrt{3}}{2} \rho^{2/3}$$

yields

$$w_{\rho\rho} = \frac{(w_\rho)^2}{w} - \frac{w_\rho}{\rho} + \frac{w^2}{\rho} - \frac{1}{w}$$

which is also Painleve III, this time with parameters $(1, 0, 0, -1)$ (it may be interesting to note that the Painleve ODEs also arise in the theory of Chern–Simons–Higgs vortices [15]). The asymptotic connection formulae for this equation have been obtained in [10]. There exists a one-parameter family of solutions singular only at the origin. Adapting the results of [10] to our case² we find

$$\begin{aligned} h(r) = 3u &\sim \left(\frac{9p}{\pi} - 6\right) \log r + \beta \quad \text{for } r \rightarrow 0 \\ &\sim \frac{6\sqrt{3}}{\pi} \left(\cos p + \frac{1}{2}\right) K_0(r) \quad \text{for } r \rightarrow \infty, \end{aligned} \quad (14)$$

where $0 < p < \pi$ parametrises the solutions,

$$\beta = 3 \ln \left(3^{-3p/\pi} \frac{9p^2}{2\pi^2} \frac{\Gamma(1 - \frac{p}{2\pi})\Gamma(1 - \frac{p}{\pi})}{\Gamma(1 + \frac{p}{2\pi})\Gamma(1 + \frac{p}{\pi})} \right) - \left(\frac{9p}{2\pi} - 3 \right) \ln 12,$$

and we have used the asymptotic formula for the Bessel function (12). Comparing this with the N -vortex asymptotics (4) we find that the range of p forces $N = 1$ in which case $p = 8\pi/9$ and the strength of the Tzitzeica one-vortex is

$$\left| \frac{3\sqrt{3}}{\pi} (2 \cos(8\pi/9) + 1) \right| \sim 1.45.$$

The resulting metric near $r = 0$ is given by

$$g = e^{-2u} (dr^2 + r^2 d\theta^2) \sim \text{const} \left(dR^2 + \frac{1}{9} R^2 d\theta^2 \right), \quad \text{where } R = r^{1/3}.$$

Thus near the origin the metric is flat, and has a conical singularity with the deficit angle $4\pi/3$.

4 Further remarks

Static vortices are solitons in two dimensions relevant in the theory of thin superconductors [1]. The Higgs field ϕ describes the density and phase of the superconducting paired electrons, coupled to an electromagnetic gauge potential a . The phase symmetry is spontaneously broken, and consequently electromagnetic field possesses a length scale. In the corresponding relativistic theory this results in the existence of massive photons. The full Ginsburg-Landau energy functional

$$E[a, \phi] = \frac{1}{2} \int_{\Sigma} \left(|D\phi|^2 + |F|^2 + \frac{c}{4} (1 - |\phi|^2)^2 \right) \text{vol}_{\Sigma}$$

²We need formulae from page 2081 with $\varepsilon = 1, \tau = r^2/12$. Formula (18) in Kitaev's paper is used with $g_1 = g_2 = 0, g_3 = 1$ and $s - 1 = 2 \cos p$ with $0 < p < \pi$ so that $\mu = 3p/2\pi$.

leads to second order field equations. The model analysed in this paper corresponds to the critical value of the coupling constant $c = 1$. The case of positive (respectively negative) c corresponds to Type I (respectively Type II) superconductors. It is known [13] that certain alloys have c arbitrary close to 1. The critical coupling is relevant from the point of view of vortex dynamics, as near the critical coupling the dynamics of slowly moving vortices can be approximated by a geodesic motion on the moduli space of static vortices, where the moduli space metric arises from the kinetic term in the vortex Lagrangian.

The solutions constructed in (13) and (14) correspond to vortices on curved backgrounds (Σ, g) . The study of such vortices is also physically interesting. For example the case where Σ is a flat torus corresponds to periodic Abrikosov lattice of vortices in Type II superconductors [1]. The general curved backgrounds can be relevant to curved thin superconducting materials in three-space.

The case of $\Sigma = \mathbb{H}^2$ with a hyperbolic metric gives rise to rotationally symmetric Yang–Mills instantons [20]. Both the Tzitzeica and Sine-Gordon vortices constructed in this paper can be considered as deformations (albeit isolated - in the sense made clear in the paper) of this example. The corresponding self-dual Yang Mills equations are satisfied on a Kahler four-manifold $\Sigma \times S^2$. The scalar-curvature of this four manifold is non-zero, so this ansatz goes beyond the analysis of integrable self-dual backgrounds.

We have shown that a radially symmetric CMC surface (Σ, g) arising from the elliptic Sinh–Gordon equation (6) can be regarded as a 1-vortex in the Abelian Higgs model. Another 1-vortex corresponds to a radial solution of an elliptic Tzitzeica equation, and thus to a hyperbolic affine sphere.

The vortex number arises as a topological invariant of the tangent bundle $T\Sigma$. This is not unlike the recent model proposed in [2], where the baryon number has been identified with the signature of the underlying four-manifold. In [2] the metric connection presumably (as the details are not given) gives rise to a gauge potential whose holonomy is the Skyrme field describing a particle. In our approach the metric connection on Σ leads directly to a vortex. There may exist other ways - see [8] for one possibility - of identifying the vortex Hermitian line bundle with a tensor power of $T\Sigma$.

The sinh–Gordon vortex, as well as the Tzitzeica vortex are isolated solutions as modifying the parameters of vortices would change the underlying Riemannian structures (Σ, g) . The existence of moduli spaces is an interesting problem which we have not been able to resolve. To understand the nature of the difficulties let us focus on the sinh–Gordon vortex (13). As we have seen, there exist two ‘obvious’ solutions to the Taubes equation on the CMC surface Σ with the metric (6): $h = 0$ and $h = -2u$. Take u to be the radially symmetric solution of the sinh–Gordon equation which leads to a 1-vortex, and consider the perturbed vortex $h = -2u + \phi$, where ϕ is small.

The linearised Taubes equation gives

$$\Delta_0 \phi = e^{-u} \phi$$

and we stress that this is not the same as the linearised sinh–Gordon equation. We require $\phi \rightarrow \text{const}$ as $r \rightarrow 0$ and $\phi \rightarrow 0$ as $r \rightarrow \infty$ to preserve the vortex number. Looking at the separable solutions gives a radial linear ODE. Near $r = 0$ this ODE admits one regular power series solution. For large r the ODE reduces to a modified Bessel equation, which admits one solution vanishing at infinity. The moduli could be constructed if we managed to relate the overall multiplicative constants in these asymptotic solutions, and establish the regularity for intermediate values of r . The metric on the underlying CMC surface is approximately flat for large and small r , but admits a conical singularity at $r = 0$ which leads to a deficit angle depending on a strength of the vortex. Thus it is not clear whether the results of Taubes [19] imply the existence of the moduli space.

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References

- [1] Abrikosov, A. A. (1957) On the magnetic properties of superconductors of the second group. Sov. Phys. JETP **5**, 1174.
- [2] Atiyah, M, Manton, N. S. and Schroers B. J. (2011) Geometric Models of Matter. [arXiv:1108.5151](#).
- [3] Brander, D., Rossman, W. and Schmitt, N. (2010) Holomorphic representation of constant mean curvature surfaces in Minkowski space: Consequences of non-compactness in loop group methods. Advances in Mathematics **223**, 949-986.
- [4] Dorfmeister, J. F., Guest, M. A., and Rossman, W. (2010) The tt^* Structure of the Quantum Cohomology of CP^1 from the Viewpoint of Differential Geometry. Asian J. Math. **14**, 417-438.
- [5] Dunajski, M. (2002) Hyper-complex four-manifolds from the Tzitzéica equation, J. Math. Phys. **43**, 651-658.
- [6] Dunajski, M. (2009) *Solitons, Instantons & Twistors*. Oxford Graduate Texts in Mathematics **19**, Oxford University Press.
- [7] Dunajski, M. and Plansangkate, P. (2009) Strominger–Yau–Zaslow geometry, Affine Spheres and Painlevé III, Comm. Math. Phys. **290**, 997-1024.

- [8] Fox, D. J. F. (2010) Einstein-like geometric structures on surfaces. [arXiv:1011.5723](#)
- [9] Jaworski, M. and Kaup, D. (1990) Direct and inverse scattering problem associated with the elliptic sinh–Gordon equation. *Inv. Prob.* **6**, 543–556.
- [10] Kitaev, A. V. (1987) Method of Isomonodromy Deformations for the ‘degenerate’ third Painlevé equation, *Zap. Nauchn. Semin. LOMI* **161**. English translation in *J. Soviet Math.* (1989) **46**, 2077–2082.
- [11] Kitaev, A. V. and Vartanian, A. (2010) Connection formulae for asymptotics of solutions of the degenerate third Painlevé equation: II. *Inverse Problems* **26** 105010.
- [12] Manton, N. S. and Rink, N. (2009) Vortices on Hyperbolic Surfaces. [arXiv:0912.2058](#)
- [13] Manton, N. S. and Sutcliffe, P. (2004) *Topological Solitons*, CUP.
- [14] McCoy, B. M., Tracy, C. A. and Wu, T. T. (1977) Painlevé functions of the third kind. *J. Math Phys.* **18**, 1058–1091.
- [15] Schiff, J. (1991) Integrability of Chern-Simons-Higgs and Abelian Higgs vortex equations in a background metric. *J. Math. Phys.* **32**, 753.
- [16] Speight J. M. (1997) Static intervortex forces. *Phys. Rev. D* **55**, 3830
- [17] Smyth, B. (1993) A generalisation of a theorem of Delaunay on constant mean curvature surfaces, *IMA Vol. Math. Appl* **51**, 123–130.
- [18] Strachan, I. A. B. (1992) Low velocity scattering of vortices in a modified Abelian Higgs model. *J. Math. Phys.* **33** 102–110
- [19] Taubes, C. H. (1980) Arbitrary N–vortex solutions to the first order Ginsburg–Landau equations. *Comm. Math. Phys.* **72**, 277–292.
- [20] Witten, E. (1977) Some Exact Multi - Instanton Solutions of Classical Yang-Mills Theory. *Phys. Rev. Lett.* **38** 121.
- [21] Yamada, K. (1988) Complete Space–Like Surfaces with Constant Mean Curvature in the Minkowski 3–Space. *Tokyo J. Math.* **11** 329–338.
- [22] Wang, C.P. (1991) Some examples of complete hyperbolic affine 2-spheres in \mathbb{R}^3 . in *Global Differential Geometry and Global Analysis* Lecture Notes in Mathematics, Vol **1481**.